¡Fun with the bell curve!

by your pal Ben

kurtosis The main question is, why is the kurtosis of a $\mathcal{N}(0,1)$ magically equal to three? Or more generally, why is

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^4 e^{\frac{-x^2}{2\sigma^2}}dx = 3\sigma^4? \tag{1}$$

I assume $\mu = 0$ for notational convenience, but the coordinate translation is trivial. We replace $-x^2$ in the exponent with $-(x-\mu)^2$, and since we are looking for a central moment, replace x^4 in equation 1 with $(x-\mu)^4$. The notation is uglier but the math will be exactly the same.

We need to apply integration by parts, wherein

$$\int vdu = u \cdot v - \int udv. \tag{2}$$

To apply the formula, set:

$$\begin{array}{cc} du = \frac{-x}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}dx & u = e^{\frac{-x^2}{2\sigma^2}} \\ v = -\sigma^2 x^3 & dv = -3\sigma^2 x^2 dx \end{array}$$

The reader will note that $v \cdot du$ is indeed the integral from equation 1. Then, applying formula 2, equation 1 becomes:

$$\frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2 x^3 e^{\frac{-x^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} + 3\sigma^2\int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{2\sigma^2}}dx\right] \tag{3}$$

The first part is symmetric around zero, and so we need not do any calculation to see that it will be zero. The integral (after multiplying through $1/\sqrt{2\pi}\sigma$) is the equation for the second moment, aka σ^2, meaning that equation 3 reduces to $3\sigma^4$, as expected.

variance Now, dear reader, you may feel that the above was begging the real question, which is: why does something that involves the square root of pi evaluate to something as nice as 'three'? To answer this, we need to explain why it is that the variance of a $\mathcal{N}(\mu,\sigma)$ is σ^2. That is, how do we evaluate the integral at the end of equation 3? We need only reapply formula 2, with:

$$\begin{array}{cc} du = \frac{-x}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}dx & u = e^{\frac{-x^2}{2\sigma^2}} \\ v = -\sigma^2 x & dv = -\sigma^2 dx \end{array}.$$

Notice that this is exactly like the first application of the formula, but with $v = -\sigma^2 x$ instead of $v = -\sigma^2 x^3$. Then, the second moment is:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2 x e^{\frac{-x^2}{2\sigma^2}}\bigg|_{-\infty}^{\infty} + \sigma^2\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\,dx\right] \tag{4}$$

The left part is again symmetric and zero by inspection. The integral on the right is a little more difficult. The trick[1] is to square the integral:

$$\left[\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\,dx\right]^2 = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\,dx \cdot \int_{-\infty}^{\infty} e^{\frac{-y^2}{2\sigma^2}}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-(x^2+y^2)}{2\sigma^2}}\,dxdy.$$

These rearrangements work because we can think of an integral as a sum, so squaring that sum can either be written as $(\sum_i a_i)\cdot(\sum_j a_j)$ or as $\sum_i\sum_j a_i \cdot a_j$.

Now rewrite the problem in terms of polar coordinates. The conversion is $x = r\cos\theta$ and $y = r\sin\theta$, meaning that $x^2 + y^2 = r$. The change-over of the integral requires multiplying by the determinant of the Jacobian; that is:

$$\begin{vmatrix} \cos\theta & \sin\theta \\ r\cdot -\sin\theta & r\cdot\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

The resulting rewritten integral and its solution are:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2\sigma^2}}\,r dr d\theta &= \int_0^{2\pi}\left[-\sigma^2\, e^{\frac{-r^2}{2\sigma^2}}\bigg|_0^{\infty}\right] d\theta \\
&= \int_0^{2\pi} \sigma^2 d\theta \\
&= 2\pi\sigma^2
\end{aligned}$$

Recall that we had squared the integral to do all of this, so the correct solution is the square root of this: $\sqrt{2\pi}\sigma$. Now, we can substitute that in to equation 4 to find the correct second moment:

$$\frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma x e^{\frac{-x^2}{2\sigma^2}}\bigg|_{-\infty}^{\infty} + \sigma^2\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\,dx\right] = 0 + \frac{1}{\sqrt{2\pi}\sigma}\cdot\sigma^2\cdot\sqrt{2\pi}\sigma = \sigma^2.$$

QEFD.

[1] Shown to me by my brother Guy; thanks.